# A Universal Turing Machine with 3 States and 9 Symbols

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**Abstract.** With an **UTM(3,9)** we present a new small universal Turing machine with 3 states and 9 symbols, improving a former result of an **UTM(3,10)**.

## 1 Introduction

After introducing the problem of the construction of simple universal Turing machines by Shannon [14], a number of small universal Turing machines have been found [5, 9]. A presentation of the records until 1996 was presented in [12], stating that there are universal Turing machines of the types **UTM(24,2)**, **UTM(10,3)**, **UTM(7,4)**, **UTM(5,5)**, **UTM(4,6)**, **UTM(3,10)**, and **UTM(2,18)**. Here **UTM(m,n)** denotes the class of universal Turing machines with $m$ states and $n$ symbols, the halting state not counted. In [13] a universal Turing machine in **UTM(22,2)** was presented. In this paper we present a machine of type **UTM(3,9)**, improving the result from [12].

The result in [13] on **UTM(22,2)**, recent results in [1] on **UTM(19,2)**, and the present result reduce the number of classes **UTM(m,n)** with an unsettled emptiness problem to 45, i.e if **UTM(m,n)** is empty.

For the basic definitions of universal Turing machines we refer to [12], and for the historical development of the subject to the bibliography.

## 2 Construction of Universal Turing Machines

We deal with the ordinary notion of a *deterministic Turing machine* (TM) with a one-dimensional tape and one head, the notion of a *configuration* of TM, and of a *tag-system* (cf. [2]). In what follows, denote via $\alpha$, $\beta$ (with or without subscripts) the configurations of a TM (tag-system or other algorithm model). Let $\beta_1 \stackrel{M}{\rightarrow} \beta_2$ mean that TM M moves from a configuration $\beta_1$ to a configuration $\beta_2$ by one step, and we write $\beta_1 \stackrel{M}{\Rightarrow} \beta_j$ for $\beta_1 \stackrel{M}{\rightarrow} \beta_2 \stackrel{M}{\rightarrow} \ldots \stackrel{M}{\rightarrow} \beta_j$.

**Definition :** A *$\mu$-tag-system* is a triple $T = (\mu, A, \Pi)$, where $\mu$ is a positive integer, the *tag-number*, $A = \{a_1, \cdots, a_n, a_{n+1}\}$ a finite alphabet, and $\Pi$ a mapping $\Pi : A \rightarrow A^* \cup \{STOP\}$ with $\Pi(a_{n+1}) = STOP$.

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The words $\alpha_i = \Pi(a_i) \in A^* \cup \{STOP\}$ are called the productions of $T$. The symbol $a_{n+1}$ is called the *halting symbol*.

A computation of a tag-system $T = (\mu, A, \Pi)$ on a word $\beta \in A^*$ is a sequence $\beta_0, \beta_1, \cdots$ such that for all $k \geq 0$ $\beta_k$ is transformed into $\beta_{k+1}$ by deleting the first $\mu$ symbols of $\beta_k$ and appending $\alpha_i$ to the result if the first symbol of $\beta_k$ is $a_i$. The computation halts if the length $|\beta_k| < \mu$ or if the first symbol of $\beta_k$ is $a_{n+1}$. We will also use the notation $a_i \rightarrow \alpha_i$ $1 \leq i \leq n$, $a_{n+1} \rightarrow STOP$, respectively.

Minsky [5] has proved the existence of universal 2-tag-systems. Such systems have been used for the construction of all small universal Turing machines so far, and will be used here too. Furthermore, he has shown that

1. The computation of a tag-system stops only if the first symbol is the halting symbol $a_{n+1}$.
2. The productions $\alpha_i$, $1 \leq i \leq n$, are not empty.

A universal Turing machine $U$ simulates a tag-system $T$ in the following way. Let $T = (2, A, \Pi)$ with $A = \{a_1, \cdots, a_n, a_{n+1}\}$ and $a_i \rightarrow \alpha_i$. Associated to each symbol $a_i \in A$ there is a $N_i > 0$, and codes $A_i$ and $\tilde{A}_i$ ( possibly $A_i = \tilde{A}_i$ as for the construction of our machine ), of the form $u^{N_i}$ with $u$ a string over the alphabet of the Turing machine $U$.

The codes $A_i$ ( $\tilde{A}_i$ ) are separated by markers on the tape of $U$.

For $1 \leq i \leq n$, the production $a_i \rightarrow \alpha_i = a_{i1} \cdots a_{im_i}$ of the tag-system $T$ is encoded by $P_i = A_{im_i} \cdots A_{i2} A_{i1}$.

The initial word $\beta = a_r a_s a_t \cdots a_w$ to be transformed by $T$, is encoded by $S = A_r A_s A_t \cdots A_w$ ( or $S = \tilde{A}_r \tilde{A}_s \tilde{A}_t \cdots \tilde{A}_w$ ).

The initial inscription on the tape of the $UTM$ $U$ is :

$Q_L P S Q_R$ with $P = P_{n+1} P_n \cdots P_1 P_0$ and $S = A_r A_s A_t \cdots A_w$,

where $Q_L$ and $Q_R$ are the infinite parts to the left and to the right of the relevant inscription on the tape of $U$, consisting only of blanks, $P_{n+1}$ is the encoding of the halting symbol $a_{n+1}$, $P_0$ an additional code, generally consisting of several markers, and the head of $U$ is located on the left of the code $S$ in state $q_1$, the initial state of $U$.

For a tag-system $T$ let $S_1$ and $S_2$ be the encodings of words $\beta_1$ and $\beta_2$, respectively, and $\beta_1 \stackrel{T}{\rightarrow} \beta_2$. Then $U$ transforms as :

$Q_L P R_1 S_1 Q_R \stackrel{U}{\Rightarrow} Q_L P R_2 S_2 Q_R$

where the rightmost two symbols of $R_2$ correspond to the cells having contained the first two symbols of $\beta_1$. For the initial tape inscription of $U$ we have $R_1 = \lambda$.

$U$ simulates the tag-system in 3 stages :

(i) In the first stage, $U$ searches the code $P_r$ corresponding to the code $A_r$, and after that $U$ deletes the codes $A_r$ and $A_s$.

(ii) In the second stage, $U$ writes the code $P_r$ at the left end of $Q_R$ as $A_{r1} \cdots A_{rm_r}$ ( note that is done in the reversed order of $P_r$ ).

(iii) In the third stage, $U$ restores its tape ( essentially the part where $P_{r-1} \cdots P_1 P_0$ is located ), and a new simulation cycle can start.

The number $N_r$ corresponding to the symbol $a_r$ ( $1 \leq r \leq n+1$ ) of the tag-system $T$, has the property that there are exactly $N_r$ markers between the codes $P_r$ and $A_r$ at each simulation cycle.

In the first simulation stage, the head of $U$ passes a number of markers in part $P$, equal to the number of symbols $x$ in code $A_r$. After this first stage of simulating, the tape of $U$ is

$Q_L P_{n+1} P_n \cdots P_{r+1} P_r P'_{r-1} \cdots P'_1 P'_0 R'_1 A'_r A_s A_t \cdots A_w Q_R$,

the head of $U$ is located between $A'_r$ and $A_s$. Then $U$ deletes $A_s$, and the second stage starts.

After the second simulation stage, the tape of $U$ is

$Q_L P_{n+1} P_n \cdots P_{r+1} P''_r P''_{r-1} \cdots P''_1 P''_0 R''_1 A_t \cdots A_w A_{r1} \cdots A_{rm_r} Q_R$,

the head of $U$ is located on the left side of $P''_r$, and the third stage starts.

After the third simulation stage, the tape of $U$ is

$Q_L P_{n+1} P_n \cdots P_1 P_0 R_2 A_t \cdots A_w A_{r1} \cdots A_{rm_r} Q_R$,

and the head of $U$ is located on the right side of $R_2$. Now a new simulation cycle can start.

Let $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ be symbols of $U$. $a_1 a_2 \cdots a_k R b_1 b_2 \cdots b_k$ means the change of the group $a_1 a_2 \cdots a_k$ into the group $b_1 b_2 \cdots b_k$ and the move of the head of $U$ from symbol $a_1$ $k$ positions to the right ( i.e. $q a_1 a_2 \cdots a_k \overset{U}{\Rightarrow} b_1 b_2 \cdots b_k q'$ where $q, q'$ are states ). Similarly, the notation $a_1 a_2 \cdots a_k L b_1 b_2 \cdots b_k$ means such a change and move to the left ( i.e. $x a_1 a_2 \cdots q a_k \overset{U}{\Rightarrow} q' x b_1 b_2 \cdots b_k$ where $x$ is an arbitrary symbol ).

group

$L a_1 a_2 \cdots a_k (b_1 b_2 \cdots b_k) R$ means the change of the group $a_1 a_2 \cdots a_k$ into the group $b_1 b_2 \cdots b_k$ and the move of the head of $U$ from $a_k$ one position to the right ( i.e. $a_1 a_2 \cdots q a_k \overset{U}{\Rightarrow} b_1 b_2 \cdots b_k q'$ ).

Similarly, $R a_1 a_2 \cdots a_k (b_1 b_2 \cdots b_k) L$ means such a change in the opposite direction ( i.e. $x q a_1 a_2 \cdots a_k \overset{U}{\Rightarrow} q' x b_1 b_2 \cdots b_k$ ).

## 3 The UTM with 3 States and 9 Symbols

The symbols of the machine in **UTM(3,9)** are $\overleftarrow{1}$ (blank symbol), 1, $\overrightarrow{1}$, $b$, $\overleftarrow{b}$, $\overrightarrow{b}$, $c$, $\overleftarrow{c}$, $\overrightarrow{c}$; and the states are – $q_1, q_2, q_3$.

$$N_1 = 1, \quad N_{k+1} = N_k + 2m_k + 2 \quad (k \in \{1, \ldots, n\}).$$

The code of the production $\alpha_i = a_{i1} a_{i2} \ldots a_{im_i}$ $(i \in \{1, \ldots, n\})$ of the tag-system is:

$$P_i = b1bbb1^{N_{im_i}} bb1^{N_{im_i - 1}} \ldots bb1^{N_{i2}} bb1^{N_{i1}},$$

where $A_j = 1^{N_j}$, $j \in \{1, \ldots, n+1\}$ and the symbol $b$ is a marker.

$$P_0 = b, \qquad P_{n+1} = \overrightarrow{1}\, b\ .$$

The code $S$ of the initial word $\beta = a_r a_s a_t \ldots a_w$, to be transformed by the tag-system, is:

$$S = c1^{N_r} c1^{N_s} c1^{N_t} \ldots c1^{N_w},$$

where the symbol $c$ is a marker.

**The program of the machine in UTM(3,9):**

| | | |
|---|---|---|
| $q_1\ b\overleftarrow{b}\ Rq_1$ | $q_2\ b\overrightarrow{b}\ Lq_3$ | $q_3\ b\overrightarrow{c}\ Rq_1$ |
| $q_1\ \overleftarrow{b}b\ Lq_1$ | $q_2\ \overleftarrow{b}\overrightarrow{b}\ Lq_2$ | $q_3\ \overleftarrow{b}\overrightarrow{b}\ Lq_3$ |
| $q_1\ \overrightarrow{b}\overleftarrow{b}\ Rq_1$ | $q_2\ \overrightarrow{b}\overleftarrow{b}\ Rq_2$ | $q_3\ \overrightarrow{b}b\ Rq_3$ |
| $q_1\ 1\overrightarrow{c}\ Lq_1$ | $q_2\ 1\overleftarrow{1}\ Rq_2$ | $q_3\ 11\ Rq_3$ |
| $q_1\ \overrightarrow{1}\overleftarrow{c}\ Rq_1$ | $q_2\ \overrightarrow{1}\overleftarrow{1}\ Rq_2$ | $q_3\ \overrightarrow{1}\ -$ |
| $q_1\ \overleftarrow{1}\overrightarrow{c}\ Lq_1$ | $q_2\ \overleftarrow{1}\overrightarrow{1}\ Lq_2$ | $q_3\ \overleftarrow{1}c\ Lq_3$ |
| $q_1\ c\overrightarrow{1}\ Lq_2$ | $q_2\ c\overleftarrow{c}\ Rq_2$ | $q_3\ c1\ Rq_1$ |
| $q_1\ \overleftarrow{c}1\ Lq_3$ | $q_2\ \overleftarrow{c}\overrightarrow{c}\ Lq_2$ | $q_3\ \overleftarrow{c}1\ Lq_3$ |
| $q_1\ \overrightarrow{c}\overleftarrow{1}\ Rq_1$ | $q_2\ \overrightarrow{c}\overleftarrow{c}\ Rq_2$ | $q_3\ \overrightarrow{c}\overrightarrow{b}\ Lq_2$ |

(i) On the first stage of modelling ( rules used in parentheses ):

$$
\begin{array}{ll}
1L\ \overrightarrow{c} & (q_1\ 1\overrightarrow{c}\ Lq_1),\\
\overleftarrow{1}\ L\ \overrightarrow{c} & (q_1\ \overleftarrow{1}\overrightarrow{c}\ Lq_1),\\
\overleftarrow{b}\ L\ b & (q_1\ \overleftarrow{b}b\ Lq_1),\\
L\ b\ (\overleftarrow{b})R & (q_1\ b\overleftarrow{b}\ Rq_1),\\
\overrightarrow{c}\ R\ \overleftarrow{1} & (q_1\ \overrightarrow{c}\overleftarrow{1}\ Rq_1),\\
b\ R\ \overleftarrow{b} & (q_1\ b\overleftarrow{b}\ Rq_1),\\
R\ 1\ (\overrightarrow{c})L & (q_1\ 1\overrightarrow{c}\ Lq_1).
\end{array}
$$

If the head of the UTM moves to the right and meets the marker $c$, the first stage of modelling is over. The UTM deletes this marker and the second stage of modelling begins ($q_1\ c\overrightarrow{1}\ Lq_2$).

(ii) On the second stage of modelling the UTM writes the markers $\overrightarrow{c}$ and the symbols $\overrightarrow{1}$ in $Q_R$; moreover the UTM writes the marker $\overrightarrow{c}$ only after the symbol $\overrightarrow{1}$.

If the UTM writes the symbol $\overrightarrow{1}$ in $Q_R$ either after writing the marker $\overrightarrow{c}$ or after the first stage of modelling, then:

$$
\begin{array}{ll}
\overrightarrow{b}\ R\ \overleftarrow{b} & (q_2\ \overrightarrow{b}\overleftarrow{b}\ Rq_2),\\
1\ R\ \overleftarrow{1} & (q_2\ 1\overleftarrow{1}\ Rq_2),\\
c\ R\ \overleftarrow{c} & (q_2\ c\overleftarrow{c}\ Rq_2),\\
R\ \overleftarrow{1}\ (\overrightarrow{1})L & (q_2\ \overleftarrow{1}\overrightarrow{1}\ Lq_2).
\end{array}
$$

If the UTM writes the symbol $\overrightarrow{1}$ in $Q_R$ after the same symbol $\overrightarrow{1}$, then:

$$
\begin{array}{ll}
\overrightarrow{b}\ R\ \overleftarrow{b} & (q_2\ \overrightarrow{b}\overleftarrow{b}\ Rq_2),\\
\overrightarrow{1}\ R\ \overleftarrow{1} & (q_2\ \overrightarrow{1}\overleftarrow{1}\ Rq_2),\\
\overrightarrow{c}\ R\ \overleftarrow{c} & (q_2\ \overrightarrow{c}\overleftarrow{c}\ Rq_2),\\
R\ \overleftarrow{1}\ (\overrightarrow{1})L & (q_2\ \overleftarrow{1}\overrightarrow{1}\ Lq_2).
\end{array}
$$

The UTM goes to the left after writing the symbol $\overrightarrow{1}$ in $Q_R$:

$$\begin{array}{ll} \overleftarrow{1}\ L\ \overrightarrow{1} & (q_2\ \overleftarrow{1}\overrightarrow{1}\ Lq_2), \\ \overleftarrow{c}\ L\ \overrightarrow{c} & (q_2\ \overleftarrow{c}\overrightarrow{c}\ Lq_2), \\ \overleftarrow{b}\ L\ \overrightarrow{b} & (q_2\ \overleftarrow{b}\overrightarrow{b}\ Lq_2). \end{array}$$

Then, if the head of the UTM meets the symbol $1$ in $P_r$, the head will change the direction of its motion, the UTM changes the symbol $1$ to the symbol $\overleftarrow{1}$ and writes the symbol $\overrightarrow{1}$ in $Q_R$:

$$L\ 1\ (\overleftarrow{1})R \quad (q_2\ 1\overleftarrow{1}\ Rq_2).$$

If the head of the UTM meets the markers $bb$ in $P_r$, then the UTM writes the marker $\overrightarrow{c}$ in $Q_R$:

$$L\ bb\ (\overrightarrow{c}\overleftarrow{b})R \quad (q_2\ b\overrightarrow{b}\ Lq_3,\ q_3\ b\overrightarrow{c}\ Rq_1,\ q_1\ \overrightarrow{b}\overleftarrow{b}\ Rq_1), \text{ then}$$

$$\begin{array}{ll} \overrightarrow{b}\ R\ \overleftarrow{b} & (q_1\ \overrightarrow{b}\overleftarrow{b}\ Rq_1), \\ \overrightarrow{1}\ R\ \overleftarrow{c} & (q_1\ \overrightarrow{1}\overleftarrow{c}\ Rq_1), \\ \overrightarrow{c}\ R\ \overleftarrow{1} & (q_1\ \overrightarrow{c}\overleftarrow{1}\ Rq_1), \\ R\ \overleftarrow{1}\ (\overrightarrow{c})L & (q_1\ \overleftarrow{1}\overrightarrow{c}\ Lq_1). \end{array}$$

When the head of the UTM moves to the left after writing the marker $\overrightarrow{c}$ in $Q_R$, then:

$$\begin{array}{ll} \overleftarrow{c}\ L\ 1 & (q_1\ \overleftarrow{c}1\ Lq_3,\ q_3\ \overleftarrow{c}1\ Lq_3), \\ \overleftarrow{1}\ L\ c & (q_3\ \overleftarrow{1}c\ Lq_3), \end{array}$$

and the UTM restores the part $S$ of the tape. Then the UTM meets the marker $\overleftarrow{b}$ in $P$:

$$\begin{array}{ll} \overleftarrow{b}\ L\ \overrightarrow{b} & (q_3\ \overleftarrow{b}\overrightarrow{b}\ Lq_3), \\ \overleftarrow{c}\ L\ 1 & (q_3\ \overleftarrow{c}1\ Lq_3), \\ \overrightarrow{c}\ L\ \overrightarrow{b} & (q_3\ \overrightarrow{c}\overrightarrow{b}\ Lq_2). \end{array}$$

The UTM halts when it meets the pair $\overrightarrow{1}b\ (q_2\ b\overrightarrow{L}\ q_3,\ \ q_3\ \overrightarrow{1}\ -)$.

If the UTM meets the pair $1b$, then the second stage of modelling is over. Then

$$L1b(1b)R \quad (q_2\ b\overrightarrow{b}\ Lq_3,\ q_3 11Rq_3,\ q_3\ \overrightarrow{b}b\ Rq_3)$$

and the third stage of modelling begins.

(iii) On the third stage of modelling the UTM restores the tape in $P$ (the tape is restored in $S$ after writing the marker $\overrightarrow{c}$ in $Q_R$):

$$\begin{array}{ll} 1R1 & (q_3 11Rq_3), \\ \overrightarrow{b}\ Rb & (q_3\ \overrightarrow{b}\ bRq_3). \end{array}$$

When the head of the UTM moves to the right and meets the marker $c$, both the third stage and the whole cycle of modelling is over. The UTM deletes the marker $c$ and a new cycle of modelling begins $(q_3 c1Rq_1)$.

## 4 Example

The following example illustrates the simulation of a 2-tag-system with alphabet $A = \{a_1, a_2, a_3, a_4\}$ and tag productions $\alpha_1 = a_2a_1a_3, \alpha_2 = a_3, \alpha_3 = a_1a_2$ by the universal Turing machine.

With $N_1 = 1, N_2 = 9, N_3 = 13, N_4 = 19$ the symbols are encoded by $c1, c1^9, c1^{13}, c1^{19}$, and the tag productions by $P_0 = b$, $P_1 = b1bbb1^{13}bb1bb1^9$, $P_2 = b1bbb1^{13}$, $P_3 = b1bbb1^9bb1$, $P_4 = \overrightarrow{1}\ b$.

The initial word $a_3a_2a_1a_3a_2$, transformed into $a_1a_3a_2a_1a_2$, gives the initial configuration

$\cdots \overleftarrow{1}\,\overrightarrow{1}\ bb1bbb1^9bb1b1bbb1^{13}b1bbb1^{13}bb1bb1^9bq_3c1^{13}c1^9c1c1^{13}c1^9\ \overrightarrow{c}\,\overleftarrow{1}\ \cdots$

or

$\cdots \overleftarrow{1}\,\overrightarrow{1}\ bb1bbb1^9bb1b1bbb1^{13}b1bbb1^{13}bb1bb1^9b1q_11^{13}c1^9c1c1^{13}c1^9c\ \overleftarrow{1}\ \cdots$

One step of the tag-system is simulated by the following sequence of the Turing machine ( not all steps are given ) :

First stage

$\cdots bq_3c1^{13}c\cdots$

$\cdots b1q_11^{13}c\cdots$

$\cdots q_1b\ \overrightarrow{c}\,\overrightarrow{c}\ 1^{12}c\cdots$

$\cdots \overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{1}\ q_11^{12}c\cdots$

$\cdots$

$\cdots bq_1b\ \overrightarrow{c}^{9}\ b\ \overrightarrow{c}\,\overrightarrow{c}\,\overrightarrow{c}\ 1^{11}c\cdots$

$\cdots b\ \overleftarrow{b}\,\overleftarrow{1}^{9}\ \overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{1}\,\overleftarrow{1}\ q_11^{11}c\cdots$

$\cdots bb1\ \overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{13}\ q_1c1^9c\cdots$

$\cdots bb1\ \overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{12}\ q_2\ \overleftarrow{1}\,\overrightarrow{1}\ 1^9c\cdots$

Second stage

$\cdots bbq_21\ \overrightarrow{b}\,\overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{1}^{13}\ \cdots\ \overrightarrow{1}^{12}\,\overrightarrow{1}\,\overrightarrow{1}\ 1^9c\cdots$

$\cdots bb\ \overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{14}\ q_21^9c\cdots$

$\cdots bb\ \overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{23}\ q_2c\cdots$

$\cdots bb\ \overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{23}\,\overleftarrow{c}\ q_2\cdots$

$\cdots bb\ \overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{23}\,\overleftarrow{c}\cdots\overleftarrow{1}^{9}\,\overleftarrow{c}\ q_2\ \overleftarrow{1}\,\overleftarrow{1}\cdots$

$\cdots bb\ \overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{1}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{b}\,\overleftarrow{1}^{13}\ \cdots\ \overleftarrow{1}^{23}\,\overleftarrow{c}\cdots\overleftarrow{1}^{9}\ q_2\ \overleftarrow{c}\,\overrightarrow{1}\,\overleftarrow{1}\cdots$

$\cdots bq_2b\ \overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{1}^{13}\ \cdots\ \overrightarrow{1}^{23}\,\overrightarrow{c}\cdots\overrightarrow{1}^{9}\,\overrightarrow{c}\,\overrightarrow{1}\,\overleftarrow{1}\cdots$

$\cdots q_3b\ \overrightarrow{b}\,\overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{1}^{13}\ \cdots\ \overrightarrow{1}^{23}\,\overrightarrow{c}\cdots\overrightarrow{1}^{9}\,\overrightarrow{c}\,\overrightarrow{1}\,\overleftarrow{1}\cdots$

$\cdots \overrightarrow{c}\ q_1\ \overrightarrow{b}\,\overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{1}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{b}\,\overrightarrow{1}^{13}\ \cdots\ \overrightarrow{1}^{23}\,\overrightarrow{c}\cdots\overrightarrow{1}^{9}\,\overrightarrow{c}\,\overrightarrow{1}\,\overleftarrow{1}\cdots$

$$\cdots \overset{\rightarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{c} \cdots \overset{\leftarrow^{23}\leftarrow}{c\ 1} \cdots \overset{\leftarrow^{9}\leftarrow\leftarrow}{c\ 1\ c\ q_1\ 1\ 1} \cdots$$

$$\cdots \overset{\rightarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{c} \cdots \overset{\leftarrow^{23}\leftarrow}{c\ 1} \cdots \overset{\leftarrow^{9}\leftarrow}{c\ 1\ q_1\ c\ 1} \cdots$$

$$\cdots \overset{\rightarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{c} \cdots \overset{\leftarrow^{23}\leftarrow}{c\ 1} \cdots \overset{\leftarrow^{9}}{c\ q_3\ 1\ 1\ c\ 1} \cdots$$

$$\cdots 1^9 q_3\ \overset{\rightarrow}{c}\overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^{13} \cdots 1^{23} c \cdots 1^9 c1\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

$$\cdots 1^8 q_2 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^{13} \cdots 1^{23} c \cdots 1^9 c1\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

$$\cdots 1^8\ \overset{\leftarrow}{1}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{1}\overset{\leftarrow}{b}\overset{\leftarrow}{1}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{1} \cdots \overset{\leftarrow^{23}\leftarrow}{1\ c} \cdots \overset{\leftarrow^{9}\leftarrow\leftarrow\leftarrow}{1\ c\ 1\ c\ q_2\ 1} \cdots$$

$$\cdots$$

$$\cdots b1bbq_2 b\ \overset{\rightarrow^{9}\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow^{13}}{1\ b\ b\ 1\ b\ 1\ b\ b\ b\ 1} \cdots \overset{\rightarrow^{23}\rightarrow}{1\ c} \cdots \overset{\rightarrow^{9}\rightarrow\rightarrow\rightarrow\rightarrow^{9}\leftarrow}{1\ c\ 1\ c\ 1\ 1} \cdots$$

$$\cdots b1bq_3 b\ \overset{\rightarrow\rightarrow^{9}\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow^{13}}{b\ 1\ b\ b\ 1\ b\ 1\ b\ b\ b\ 1} \cdots \overset{\rightarrow^{23}\rightarrow}{1\ c} \cdots \overset{\rightarrow^{9}\rightarrow\rightarrow\rightarrow\rightarrow^{9}\leftarrow}{1\ c\ 1\ c\ 1\ 1} \cdots$$

$$\cdots b1b\ \overset{\rightarrow}{c}\ q_1\ \overset{\rightarrow\rightarrow^{9}\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow^{13}}{b\ 1\ b\ b\ 1\ b\ 1\ b\ b\ b\ 1} \cdots \overset{\rightarrow^{23}\rightarrow}{1\ c} \cdots \overset{\rightarrow^{9}\rightarrow\rightarrow\rightarrow\rightarrow^{9}\leftarrow}{1\ c\ 1\ c\ 1\ 1} \cdots$$

$$\cdots b1b\ \overset{\rightarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow^{9}}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{c} \cdots \overset{\leftarrow^{23}\leftarrow}{c\ 1} \cdots \overset{\leftarrow^{9}\leftarrow\leftarrow\leftarrow\leftarrow^{9}}{c\ 1\ c\ 1\ c\ q_1\ 1\ 1} \cdots$$

$$\cdots b1b\ \overset{\rightarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow^{9}}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{c} \cdots \overset{\leftarrow^{23}\leftarrow}{c\ 1} \cdots \overset{\leftarrow^{9}\leftarrow\leftarrow\leftarrow\leftarrow^{8}}{c\ 1\ c\ 1\ c\ q_1\ c\ 1} \cdots$$

$$\cdots b1b\ \overset{\rightarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow^{9}}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{c}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow}{b}\overset{\leftarrow^{13}}{c} \cdots \overset{\leftarrow^{23}\leftarrow}{c\ 1} \cdots \overset{\leftarrow^{9}\leftarrow\leftarrow\leftarrow\leftarrow^{7}}{c\ 1\ c\ 1\ c\ q_3\ c\ 1\ c\ 1} \cdots$$

$$\cdots b1bq_3\ \overset{\rightarrow}{c}\overset{\rightarrow}{b}\ 1^9\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^{13} \cdots 1^{23} c \cdots 1^9 c1c1^9\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

$$\cdots b1q_2 b\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^9\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^{13} \cdots 1^{23} c \cdots 1^9 c1c1^9\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

$$\cdots bq_3 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^9\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^{13} \cdots 1^{23} c \cdots 1^9 c1c1^9\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

Third stage

$$\cdots b1q_3\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^9\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\ 1\ \overset{\rightarrow}{b}\overset{\rightarrow}{b}\overset{\rightarrow}{b}\ 1^{13} \cdots 1^{23} c \cdots 1^9 c1c1^9\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

$$\cdots b1bbb1^9 bb1b1bbb1^{13} \cdots bb1^9 b1^{23} q_3 c1c \cdots 1^9 c1c1^9\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

$$\cdots b1bbb1^9 bb1b1bbb1^{13} \cdots bb1^9 b1^{24} q_1 1c \cdots 1^9 c1c1^9\ \overset{\rightarrow}{c}\overset{\leftarrow}{1} \cdots$$

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